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Asymptotic behaviour of solutions for the wave equation with an effective dissipation around the boundary

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Abstract

We consider the initial–boundary value problem for the wave equation with a dissipation $a(t,x)u_t$ in an exterior domain, whose boundary meets no geometrical condition. We assume that the dissipation $a(t,x)u_t$ is effective around the boundary and $a(t,x)$ decays as $|x| \to \infty$. We shall prove that the total energy does not in general decay, and the solution is asymptotically free as the time goes to infinity. Further, we shall show that the local energy decays like $O(t^{-1})$ $(t \to \infty)$.

1. Introduction

In this paper we are concerned with the following initial–boundary value problem:

$$\text{(P)} \qquad \begin{cases} u_{tt} - \Delta u + a(t,x)u_t = 0, & (t,x) \in (0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \end{cases}$$

where Ω is an exterior domain in $\mathbb{R}^N$ outside a compact obstacle $\mathcal{O}$ with a smooth boundary $\partial\Omega$. For simplicity we assume that $0 \notin \overline{\Omega}$ and $\partial\Omega$ is contained in the

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ball $B_{\rho_0} \equiv \{x \in \mathbb{R}^N;\ |x| \leqslant \rho_0\}$. Further, the function $a(t,x)$ is nonnegative in $[0,\infty) \times \overline{\Omega}$, effective near the boundary $\partial\Omega$ and decays as $|x| \to \infty$.

As is well known, if $\{u_0, u_1\}$ belongs to $H_0^1(\Omega) \times L^2(\Omega)$, the problem (P) has a finite energy solution satisfying the energy identity

$$\|u(t)\|_E^2 + \int_s^t \int_\Omega a u_t^2 \, dx \, d\tau = \|u(s)\|_E^2 \quad (0 \leqslant s < t), \tag{1.1}$$

where we set

$$\|u(t)\|_E^2 = \frac{1}{2} \int_\Omega \left\{ u_t(t,x)^2 + |\nabla u(t,x)|^2 \right\} dx.$$

Since $a(t,x) \geqslant 0$ in $[0,\infty) \times \overline{\Omega}$, the total energy decreases in $t > 0$, which leads to a question whether it decays or not as $t \to \infty$.

Concerning the energy decay, when the data has a compact support and the dissipation au_t satisfies $a_0(1+r+t)^{-1} \leqslant a(t,x) \leqslant a_1$ $(r = |x|)$ and $a_t(t,x) \leqslant 0$ for some $a_i > 0$ $(i = 0,1)$, Matsumura [6] proved that the energy of the solution to the Cauchy problem in $\mathbb{R}^N$ decays like $O(t^{-1})$ as $t \to \infty$, and Mochizuki [8] generalized this result to the noncompact support data. Recently, Mochizuki and Nakazawa [11] has generalized these results to the exterior problems; that is, if $a(t,x)$ is bounded from below by some function of the inverse logarithmic order in t and x, then the energy for the solution with the logarithmically weighted initial data decays like $O(\{\log(e+t)\}^{-\mu})$, $\mu = \min\{1, a_0/2\}$, as $t \to \infty$. We should also mention the energy decay for the wave equations with the more delicate dissipation. For example, Zuazua [20] has proved the exponential decay for the semilinear wave equations with the localized dissipation near infinity, and Lagnese [5] investigated the exponential decay for the wave equations with the boundary dissipation in a bounded domain and so on. In this paper we are concerned with the converse situation, that is, we want to notice the case where the energy never decays as $t \to \infty$.

Contrary to the energy decay, Mochizuki and Nakazawa has also proved in [11] that if $\mathcal{O}$ is star-shaped $(N \geqslant 3)$ and $a(t,x)$ decays like the inverse logarithmic order (see Hypothesis A below) as $|x| \to \infty$, then the energy does not in general decay, and the finite energy solution is asymptotically free as $t \to \infty$. There are also recent works of Mochizuki [9] and Nakazawa [15] which treat the scattering problem for the Cauchy problem in $\mathbb{R}^N$ $(N \geqslant 3)$ and $\mathbb{R}^2$, respectively. Their argument is essentially based on the integrability of the local energy. However, concerning the energy nondecay to the problem (P), one knows no result except for [11]. Especially they proposed the geometrical condition on the boundary. In this paper we first establish the energy nondecay and the existence of a scattering state "*without any restriction on the shape of $\mathcal{O}$.*" Here, the energy nondecay problem is meant by constructing a solution to the problem (P) with some special data whose energy never decays. As for the another approach to the energy nondecay problem, one can find the paper of Rauch and Taylor [17]. For the nonlinear dissipative case in $\mathbb{R}^N$ $(N \geqslant 2)$ we should refer to Mochizuki and Motai [10]. Secondly, we prove that the local energy decays to 0 as $t \to \infty$. Needless to say, the more the dissipation is effect, the less the energy nondecay occurs and the local energy decays.

Observing the proof in [7,8,11] based on the weighted energy method, we can see that the integrability of the local energy plays an important role in deriving the asymptotics for the Cauchy or exterior problems, and in particular, *the geometric condition on the shape of the obstacle* $\mathcal{O}$ is essential in the exterior problem. For example, with regard to the free wave equation in nontrapping domains, Morawetz et al. [13] have obtained the integrability of the local energy, and Filinovskii [2] investigated it in domains outside the star-shaped obstacle. On the other hand, when the domain Ω has no restriction on the shape, it is difficult to discuss the asymptotics for the problem (P). The reason is that when $\mathcal{O}$ is trapping the local energy never decays uniformly (Ralston [16]), and hence it is difficult to obtain the integrability of the local energy. In this paper we want to treat "*general exterior domains*" under the effect of the dissipation around the boundary $\partial\Omega$. We shall prove the energy nondecay problem, the existence of a scattering state (Theorem 1) and the local energy decay for the problem (P) (Theorem 2). For our purpose, combining with the multiplier method (cf. Nakao [14] and Zuazua [20]), we shall use the weighted energy method to obtain the integrability of the local energy (Proposition 4.2). Our crucial idea of the proof of Theorem 1 lies in introducing a hypersurface $\widetilde{\Gamma}$ (see Fig. 2) between the trapping and nontrapping regions and obtaining the integrability of $\|u_t(t)\|^2_{L^2(\widetilde{\Gamma})}$ in Proposition 5.2. Here we must remark that our argument would go well even if the domain Ω is an exterior domain outside a finite number of bounded obstacles. Moreover, if, in particular, $\mathcal{O}$ is star-shaped, then $\widetilde{\Gamma} = \emptyset$, and our result coincides with Mochizuki and Nakazawa [11].

In order to state the assumption on $a(t,x)$, we define the positive numbers e_n and the functions $\log^{[n]}$ $(n = 0, 1, 2, \ldots)$ by

$$e_0 = 1, \quad e_1 = e, \quad \ldots, \quad e_n = e^{e_{n-1}},$$
$$\log^{[0]} a = a, \quad \log^{[1]} a = \log a, \quad \ldots, \quad \log^{[n]} a = \log\log^{[n-1]} a.$$

Then we make the following hypotheses on $a(t,x)$.

Hypothesis A. *$a(t,x)$ is a nonnegative function in $[0,\infty) \times \overline{\Omega}$ and belongs to $C^1([0,\infty) \times \overline{\Omega})$ satisfying*

$$0 \leqslant a(t,x) \leqslant a_1 \Big\{ (e_n + r)\log(e_n + r)\log^{[1]}(e_n + r)\ldots \times \log^{[n-1]}(e_n + r)\big[\log^{[n]}(e_n + r)\big]^\delta \Big\}^{-1} \tag{1.2}$$

in $[0,\infty) \times \overline{\Omega}$ for some $a_1 > 0$, $n \in \mathbb{N} \cup \{0\}$ and $\delta > 1$.

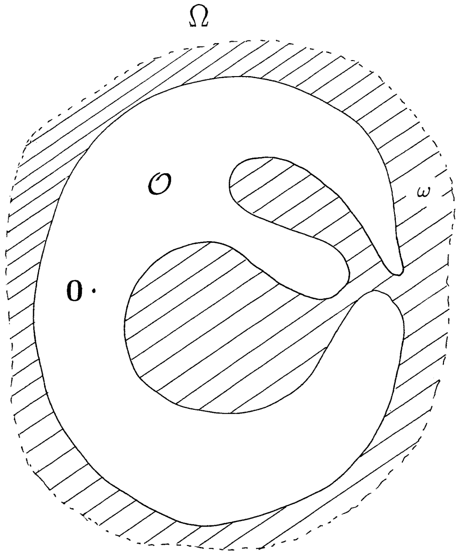

Fig. 1.

(i) *There exists a bounded and relatively open set* ω *in* $\overline{\Omega}$ *such that*

$$\partial\Omega \subset \omega \quad \text{and} \quad a(t,x) \geqslant a_0 \quad \text{in } [0,\infty)\times\omega \tag{1.3}$$

for some $a_0 > 0$.

(ii) *There exists a constant* $C > 0$ *such that*

$$|\nabla a(t,x)| \leqslant Ca(t,x) \quad \text{in } [0,\infty)\times\overline{\Omega}. \tag{1.4}$$

Once the energy nondecay was established, it would be quite natural to consider the local energy decay and determine its rate. Intuitively speaking, the concept of the local energy decay is equivalent to the phenomenon that the waves escape from the obstacle. For the local energy decay of solutions to the free wave equation, it is well known [12] that if the obstacle is star-shaped, local energy decays like $E_{\mathrm{loc}}(t) = O(t^{-1})$ $(t \to \infty)$, and in particular, it decays exponentially if the space dimension is odd integer greater than or equal to three. In more general nontrapping domains, Morawetz et al. [13] has proved the same result of [12], and in [18] Shibata and Tsutsumi has obtained the sharp decay rate. Furthermore, Ikawa [3] treated the decay in the exterior domain outside the two convex obstacles. We should also refer to the work of Tamura [19] which treated the wave equations in $\mathbb{R}^3$ with compactly supported time-dependent dissipative terms and was proved that the local energy decays exponentially. Moreover, Dan and Shibata [1] has treated the dissipative wave equation and proved $E_{\mathrm{loc}}(t) = O(t^{-N})$ $(t \to \infty)$. Quite recently, when Ω is the general domain, Nakao [14]

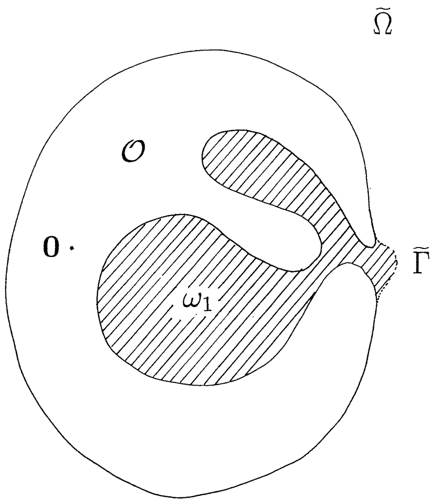

Fig. 2.

has succeeded in deriving the local energy decay for the wave equation with a localized dissipation ($\operatorname{supp} a \subset B_L$), which is effective only on a part of the boundary; that is, for the local energy $E^{\varepsilon}_{\mathrm{loc}}(t) = \frac{1}{2}\int_{\Omega\cap B_{L+\varepsilon t}}\{u_t(t)^2 + |\nabla u(t)|^2\}\,dx$, we have

$$E^{\varepsilon}_{\mathrm{loc}}(t) \leqslant C_{\varepsilon,\delta}\|u(0)\|^2_E(1+t)^{-1+\delta} \quad \text{for } 0 < \forall\varepsilon, \forall\delta < 1.$$

In this paper we shall prove in Theorem 2 that the local energy decays like $O(t^{-1})$ $(t \to \infty)$ "*without any compact support condition of a*." It should be noted that this improves a part of the result in [14]. Of course, it is easy to see that the definition of $E^{\varepsilon}_{\mathrm{loc}}(t)$ in [14] coincides with that of the present one. For this purpose, we shall derive the space–time estimate of $ar|\nabla u|^2$ in Lemma 6.1, which is essential in our argument.

2. Statement of results

Hereafter, we set

$$X^2(D) = C\big([0,\infty); H^2(D)\cap H^1_0(D)\big)\cap C^1\big([0,\infty); H^1_0(D)\big) \\ \cap C^2\big([0,\infty); L^2(D)\big)$$

for any domain D in $\mathbb{R}^N$. We make the following hypothesis on the initial data $\{u_0, u_1\}$.

Hypothesis B. *The initial data $\{u_0, u_1\}$ belongs to $[H^2(\Omega)\cap H^1_0(\Omega)]\times H^1_0(\Omega)$.*

If we assume Hypotheses A and B, then the problem (P) has a unique solution $u(t,x)$ in the class $X^2(\Omega)$ satisfying the energy identity (1.1).

Now we want to treat the free wave equation whose solutions are possibly asymptotic states of solutions to the problem (P). For this, we must restrict the domain Ω to some subdomain. Let ω_1 be a bounded open set of Ω, which contains the trapping regions of Ω, and let $\widetilde{\Gamma} \equiv \partial\omega_1 \setminus (\partial\Omega \cap \partial\omega_1)$ and we set $\widetilde{\Omega} \equiv \Omega \setminus (\omega_1 \cup \widetilde{\Gamma})$ such that $\partial\widetilde{\Omega}$ is sufficiently smooth and $\mathbb{R}^N \setminus \widetilde{\Omega}$ is star-shaped with respect to the origin. We assume $\partial\widetilde{\Omega} \subset \omega$. Let $\tilde{\boldsymbol{\nu}}(x)$ be the unit outward normal vector at $x \in \partial\widetilde{\Omega}$. Of course, if $\mathcal{O}$ is star-shaped with respect to the origin, $\widetilde{\Gamma}$ becomes the empty set and $\widetilde{\Omega} = \Omega$. Then we consider the initial–boundary value problem for the free wave equation in $\widetilde{\Omega}$ with the same initial data as the problem (P):

$$
(\mathrm{P})_0 \quad \begin{cases} w_{tt} - \Delta w = 0, & (t,x) \in (0,\infty) \times \widetilde{\Omega}, \\ w(0,x) = u_0(x), \quad w_t(0,x) = u_1(x), & x \in \widetilde{\Omega}, \\ w(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\widetilde{\Omega}. \end{cases}
$$

As is well known, if $\{u_0,u_1\} \in [H^2(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})] \times H_0^1(\widetilde{\Omega})$, then the problem $(\mathrm{P})_0$ has a unique solution $w(t,x)$ in the class $X^2(\widetilde{\Omega})$ satisfying the energy conservation law

$$
\|w(t)\|_{\widetilde{E}} = \|w(0)\|_{\widetilde{E}} = \|u(0)\|_{\widetilde{E}}, \tag{2.1}
$$

where we set

$$
\|w(t)\|_{\widetilde{E}}^2 = \frac{1}{2}\int_{\widetilde{\Omega}} \left\{ w_t(t,x)^2 + |\nabla w(t,x)|^2 \right\} dx.
$$

Thus we further impose the initial data $\{u_0,u_1\}$ as follows.

Hypothesis C. *The initial data* $\{u_0,u_1\}$ *of the problem* $(\mathrm{P})_0$ *satisfies*

$$
\{u_0,u_1\} \in \left[H^2(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})\right] \times H_0^1(\widetilde{\Omega}) \quad \textit{and} \quad \{u_0,u_1\} \neq \{0,0\} \quad \textit{in } \widetilde{\Omega}.
$$

Let us recall that ω_1 contains the trapping regions. Since $a(t,x) \geqslant a_0 > 0$ in $\omega \cap \omega_1$, it follows from the standard weighted energy method (cf. [8] and [11]) that the energy of the trapped wave $u(t,x)$ in ω_1 decays like $O(t^{-1})$ $(t \to \infty)$. The following theorem implies that some wave $u(t,x)$ escaping from ω_1 does not in general decay, and is asymptotic to the free wave $w(t,x)$ in $\widetilde{\Omega}$. Hence the wave $u(t,x)$ consists of two types; decaying state in ω_1 and scattering in $\widetilde{\Omega}$. In order to state this result, we need the estimate

$$
\int_0^\infty \int_{\widetilde{\Omega}} a w_t^2 \, dx \, d\tau + \int_0^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right)^2 dS \, d\tau \leqslant C\|w(0)\|_{\widetilde{E}}^2
$$

for some $C > 0$ independent of t and $\{u_0,u_1\}$, which will be shown in Lemma 5.1. Then, based on this estimate, we have the following.

Theorem 1. *Let* $N \geqslant 3$. *Assume Hypotheses* A *and* B. *Then we have the following assertions*:

(i) *Suppose Hypothesis* C. *Let* $w(t,x)$ *be the* $X^2(\widetilde{\Omega})$*-solution to the problem* $(\mathrm{P})_0$ *and* $\widetilde{w}(\sigma,x)$ *a zero extension of* $w(\sigma,x)$ *to* Ω, *where* $\sigma \equiv \sigma(u_0,u_1) > 0$ *is a time so that the following inequality*

$$\int_\sigma^\infty \int_{\widetilde{\Omega}} a w_t^2 \, dx \, dt + \frac{C_0\{\|u_t(0)\|_{\widetilde{E}} + \|u(0)\|_{\widetilde{E}}\}^2}{\|u(0)\|_{\widetilde{E}}^2} \int_\sigma^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right)^2 dS \, dt$$
$$< 2\|u(0)\|_{\widetilde{E}}^2 \tag{2.2}$$

holds and the constant C_0 *is independent of* σ. *Then the energy* $\|u^{(\sigma)}(t)\|_E$ *of* $u^{(\sigma)}(t,x)$ *never decays as* $t \to \infty$, *where* $u^{(\sigma)}(t,x)$ *is the finite energy solution to the problem* (P) *with the initial condition* $\{u^{(\sigma)}(0), u_t^{(\sigma)}(0)\} = \{\widetilde{w}(\sigma), \widetilde{w}_t(\sigma)\}$.

(ii) *For the* $X^2(\Omega)$*-solution* $u(t,x)$ *to the problem* (P) *with* $\|\{u_0,u_1\}\|_{H^2 \times H_0^1} < \infty$, *there exists a pair* $\{w_0^+, w_1^+\} \in \widetilde{E}$ *such that*

$$\left\|u(t) - w^+(t)\right\|_{\widetilde{E}} \to 0 \quad \text{as } t \to \infty, \tag{2.3}$$

where $w^+(t,x)$ *is the solution to the problem* $(\mathrm{P})_0$ *with* $\{u_0,u_1\}$ *replaced by* $\{w_0^+, w_1^+\}$.

Remark. If, in particular, the obstacle $\mathcal{O}$ is star-shaped with respect to the origin, $\widetilde{\Gamma}$ becomes the empty set. Hence we need not take account of Hypothesis A(i) in Theorem 1, and as a result, Theorem 1 is a natural extension of the result of Mochizuki and Nakazawa [11].

In order to state our second theorem, we must restrict the assumptions in the previous theorem to a stronger condition. If we assume the initial data $\{u_0,u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$ has a compact support, that is,

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega(R) \equiv \Omega \cap B_R \tag{2.4}$$

for $\forall R > \rho_0$, then the problem (P) has a finite energy solution $u(t,x)$ satisfying the finite propagation property

$$\operatorname{supp} u(t,\cdot) \subset \Omega(R+t). \tag{2.5}$$

Furthermore, we assume that $a(t,x)$ in Hypothesis A is independent of t and belongs to $L^\infty(\Omega)$ so that

$$0 \leqslant a(x) \leqslant a_1 \Big\{ (e_n + r)^2 \log(e_n + r) \log^{[1]}(e_n + r) \ldots$$
$$\times \log^{[n-1]}(e_n + r) \big[\log^{[n]}(e_n + r)\big]^\delta \Big\}^{-1} \tag{2.6}$$

for some $a_1 > 0$ and $\delta > 1$ with $a_1 < \delta^{-1}(\delta - 1)$.

Our second result reads as follows.

Theorem 2. *Let $N \geqslant 3$ and R be any fixed number with $R > \rho_0$. If we assume that the initial data $\{u_0, u_1\}$ satisfies the condition of the compact support as in* (2.4)*, then, under Hypothesis* A(i) *with the assumption* (2.6)*, the local energy of the finite energy solution $u(t,x)$ decays to* 0 *as $t \to \infty$, that is, we have*

$$\frac{1}{2}\int\limits_{\Omega(R)} \left\{|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right\} dx \leqslant C(R)\|u(0)\|_E^2(1+t)^{-1} \tag{2.7}$$

for all $t \geqslant 0$.

Remark. (i) Theorem 2 is partly an extension of the result in Nakao [14], which imposes the compact support condition of $a(x)$, in the following sense; if $N \geqslant 3$, then the decay rate of the local energy is sharper than $O(t^{-1+\delta})$ $(t \to \infty)$ for $0 < \forall\delta < 1$. Of course, if we set $R = L/(1-\varepsilon)$ for $t > R$, then the local energy in Theorem 2 coincides with $E^{\varepsilon}_{\mathrm{loc}}(t)$ in [14].

(ii) The reason why a should be independent of t is that we need the boundedness of $\|u(t)\|$ to control the boundary integral of $\partial u/\partial \boldsymbol{\nu}$ on a part of the boundary $\partial\Omega$, $\boldsymbol{\nu}$ being the unit outward normal vector on $\partial\Omega$. When a depends on t, it is still unknown whether $\|u(t)\|$ is bounded or not. Based on this surface integral, we can obtain the space–time estimate of $ar|\nabla u|^2$ (see Proposition 4.2 and Lemma 6.1).

3. Some formulas

In this section we summarize some identities used later.

Lemma 3.1 (Zuazua [20]). *Let $u(t,x)$ be a smooth solution to the problem* (P). *Then we have*

$$\begin{aligned}&\int\limits_0^t\int\limits_{\Omega}\left\{\frac{1}{2}(\nabla\cdot\boldsymbol{h})\big(u_t^2 - |\nabla u|^2\big) + \sum_{i,j=1}^N \frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right\} dx\,d\tau \\ &\qquad + \int\limits_0^t\int\limits_{\Omega} a(\boldsymbol{h}\cdot\nabla u)u_t\,dx\,d\tau \\ &\quad = -\big(u_t(\tau), \boldsymbol{h}\cdot\nabla u(\tau)\big)\big|_0^t + \frac{1}{2}\int\limits_0^t\int\limits_{\partial\Omega}\big\{\boldsymbol{h}(x)\cdot\boldsymbol{\nu}(x)\big\}\left(\frac{\partial u}{\partial\boldsymbol{\nu}}\right)^2 dS\,d\tau\end{aligned} \tag{3.1}$$

for any C^1-vector field $\boldsymbol{h}(x) = (h_1(x), h_2(x), \ldots, h_N(x))$, where $\boldsymbol{\nu}(x)$ is the outward normal vector at $x \in \partial\Omega$, and

$$\int_0^t \int_\Omega \eta\big(|\nabla u|^2 - u_t^2\big)\,dx\,d\tau + \int_0^t \int_\Omega (\nabla\eta\cdot\nabla u)u\,dx\,d\tau$$
$$= -\big(\eta u(\tau), u_t(\tau)\big)\big|_0^t - \int_0^t \int_\Omega \eta a u_t u\,dx\,d\tau \tag{3.2}$$

for any $\eta \in W^{1,\infty}(\Omega)$.

Proof. Multiplying the equation by $\boldsymbol{h}\cdot\nabla u$ and ηu, respectively, we have the identities (3.1) and (3.2). □

Lemma 3.2. *Let* $u(t,x)$ *be as above and* R *any fixed number with* $R > \rho_0$. *Then we have*

$$E_R(t) + \frac{N-1}{4R}\int_{\partial B_R} u^2\,dS_R$$
$$= \frac{1}{2}\int_{\Omega(R)} \left\{u_t^2 + \left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 + \frac{(N-1)(N-3)}{4r^2}u^2\right\}dx, \tag{3.3}$$

where we set

$$E_R(t) = \frac{1}{2}\int_{\Omega(R)} \left\{|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right\}dx.$$

Proof. We see

$$\left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 + \frac{(N-1)(N-3)}{4r^2}u^2$$
$$= |\nabla u|^2 + \nabla\cdot\left(\frac{N-1}{2r}\frac{x}{r}u^2\right). \tag{3.4}$$

Since $u|_{\partial\Omega} = 0$, integration of the identity (3.4) over $\Omega(R)$ leads to the identity (3.3). □

4. Space–time estimates

In this section we shall prove the space–time estimate which will be used later. First, we shall prove the space–time estimate of $a|\nabla u_t|^2$, which will be needed in Proposition 5.2.

Proposition 4.1. *Assume Hypothesis* B *and* $|\nabla a| \leqslant Ca$ *for some* $C > 0$. *Let* $u(t,x)$ *be the solution in the class* $X^2(\Omega)$ *to the problem* (P). *Then there exists a constant* $C_1 > 0$ *such that*

$$\int_0^\infty \int_\Omega a|\nabla u_t|^2\,dx\,d\tau \leqslant C_1\bigl\{\|u(0)\|_E + \|u_t(0)\|_E\bigr\}^2, \tag{4.1}$$

where we set

$$\|u_t(0)\|_E^2 = \frac{1}{2}\int_\Omega \bigl\{|u_{tt}(t,x)|^2 + |\nabla u_t(t,x)|^2\bigr\}\,dx.$$

Proof. The proof is very standard. Multiplying the equation by $-\Delta u_t$ and integrating, we have

$$\begin{aligned}
&\frac{1}{2}\bigl\{\|\nabla u_t(t)\|^2 + \|\Delta u(t)\|^2\bigr\} + \int_0^t \int_\Omega a|\nabla u_t|^2\,dx\,d\tau \\
&\quad\leqslant \frac{1}{2}\bigl\{\|\nabla u_t(0)\|^2 + \|\Delta u(0)\|^2\bigr\} + \int_0^t \int_\Omega |\nabla a||u_t||\nabla u_t|\,dx\,d\tau \\
&\quad\leqslant \frac{1}{2}\bigl\{\|\nabla u_t(0)\|^2 + \|\Delta u(0)\|^2\bigr\} + C_\varepsilon \int_0^t \int_\Omega a u_t^2\,dx\,d\tau \\
&\qquad + \varepsilon \int_0^t \int_\Omega a|\nabla u_t|^2\,dx\,d\tau
\end{aligned} \tag{4.2}$$

for $0 < \forall\varepsilon < 1$ and all $t \geqslant 0$, where we have used $|\nabla a| \leqslant Ca$. Hence, it follows from the energy identity (1.1) and the estimate (4.2) that

$$\int_0^t \int_\Omega a|\nabla u_t|^2\,dx\,d\tau \leqslant C\bigl\{\|u(0)\|_E^2 + \|\nabla u_t(0)\|^2 + \|\Delta u(0)\|^2\bigr\}. \tag{4.3}$$

Here, using the equation, we see

$$\|\Delta u(0)\|^2 \leqslant \|u_{tt}(0)\|^2 + \|a u_t(0)\|^2 \leqslant C\bigl\{\|u(0)\|_E^2 + \|u_t(0)\|_E^2\bigr\}. \tag{4.4}$$

Thus, we conclude from the estimates (4.3) and (4.4) that the estimate (4.1) is valid. □

Next, we shall give the estimate used in deriving the local energy decay.

Proposition 4.2. *Let us assume $N \geqslant 3$ and Hypothesis* A *with $0 < a_1 < \delta^{-1} \times (\delta - 1)$, and further suppose that $a(t,x)$ is independent of t. If, in addition, we assume that the initial data $\{u_0, u_1\}$ satisfies*

$$\|au_0 + u_1\|_{L^{2,1}}^2 \equiv \int_\Omega r^2 (au_0 + u_1)^2 \, dx < \infty, \tag{4.5}$$

then, for the finite energy solution $u(t,x)$ to the problem (P) *with $\|u(0)\|_E < \infty$, there exist a constant $C_2 > 0$ such that*

$$\begin{aligned}
&\int_0^\infty \int_\Omega \Big\{ (e_n + r) \log(e_n + r) \log^{[1]}(e_n + r) \ldots \\
&\qquad \times \log^{[n-1]}(e_n + r) \big[\log^{[n]}(e_n + r)\big]^\delta \Big\}^{-1} \\
&\qquad \times \left\{ u_t^2 + \left| \nabla u + \frac{N-1}{2r} \frac{x}{r} u \right|^2 + \frac{(N-1)(N-3)}{4r^2} u^2 \right\} dx \, dt \\
&\quad \leqslant C_2 \big\{ \|\{u_0, u_1\}\|_{H^1 \times L^2}^2 + \|au_0 + u_1\|_{L^{2,1}}^2 \big\}.
\end{aligned} \tag{4.6}$$

If $\mathcal{O}$ is star-shaped with respect the origin, then we do not need the assumption (4.5) *and can neglect the terms $\|u_0\|^2$ and $\|au_0 + u_1\|_{L^{2,1}}^2$ in* (4.6).

In particular, for any fixed number R with $R > \rho_0$, there exists a constant $C_3(R) > 0$ such that

$$\int_0^\infty E_R(t) \, dt \leqslant C_3(R) \big\{ \|\{u_0, u_1\}\|_{H^1 \times L^2}^2 + \|au_0 + u_1\|_{L^{2,1}}^2 \big\}, \tag{4.7}$$

where we set

$$E_R(t) = \frac{1}{2} \int_{\Omega(R)} \big\{ |u_t(t,x)^2| + |\nabla u(t,x)|^2 \big\} \, dx.$$

For the proof of Proposition 4.2, let us introduce the positive smooth function $\psi(r)$ of $r \geqslant 0$ satisfying

$$\psi(r) \text{ is bounded, monotone increasing in } r \text{ and } \psi(r) \geqslant r\psi'(r). \tag{4.8}$$

We need the following two lemmas.

Lemma 4.3. *Let $N \geqslant 3$ and $u(t,x)$ be the finite energy solution to the problem* (P) *with $\|u(0)\|_E < \infty$. Then there exists a constant $C_4 > 0$ such that*

$$\int_0^t \int_\Omega \psi'(r) \left\{ u_t^2 + \left| \nabla u + \frac{N-1}{2r} \frac{x}{r} u \right|^2 + \frac{(N-1)(N-3)}{4r^2} u^2 \right\} dx \, d\tau$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega \psi a u_t\left(u_r+\frac{N-1}{2r}u\right)dx\,d\tau \\
&\leqslant C_4\left(\|u(0)\|_E^2+\int_0^t\int_\Omega a u^2\,dx\,d\tau\right) \qquad (4.9)
\end{aligned}
$$

for all $t>0$. *If* $\mathcal{O}$ *is star-shaped with respect the origin, then the right-hand side of* (4.9) *can be replaced by* $C_4\|u(0)\|_E^2$.

Proof. We may assume that the solution $u(t,x)$ is smooth enough, because this solution is given as a limit of approximate solutions.

First step. Multiplying the equation in (P) by $\psi(r)(u_r+(N-1)/(2r)u)$, we have

$$
X_t+\nabla\cdot\boldsymbol{Y}+Z=0, \qquad (4.10)
$$

where

$$
\begin{aligned}
X&=\psi u_t\left(u_r+\frac{N-1}{2r}u\right),\\
\boldsymbol{Y}&=-\frac{1}{2}\psi\left\{\frac{x}{r}\left(u_t^2-|\nabla u|^2+\frac{N-1}{2r^2}u^2\right)+2\nabla u\left(u_r+\frac{N-1}{2r}u\right)\right\},\\
Z&=\psi a u_t\left(u_r+\frac{N-1}{2r}u\right)\\
&\quad+(r^{-1}\psi-\psi')\left\{|\nabla u|^2-u_r^2+\frac{(N-1)(N-3)}{4r^2}u^2\right\}\\
&\quad+\frac{1}{2}\psi'\left\{u_t^2+\left|\nabla u+\frac{N-1}{2r}\frac{x}{r}u\right|^2+\frac{(N-1)(N-3)}{4r^2}u^2\right\}.
\end{aligned}
$$

Integrate by parts the identity (4.10). Then, since $N\geqslant 3$ and $r^{-1}\psi-\psi'\geqslant 0$, we have

$$
\begin{aligned}
&\int_\Omega X(\tau,x)\,dx\bigg|_{\tau=0}^{\tau=t}+\int_0^t\int_{\partial\Omega}\boldsymbol{Y}\cdot\boldsymbol{\nu}\,dS\,d\tau\\
&\quad+\int_0^t\int_\Omega \psi a u_t\left(u_r+\frac{N-1}{2r}u\right)dx\,d\tau\\
&\quad+\frac{1}{2}\int_0^t\int_\Omega \psi'\left\{u_t^2+\left|\nabla u+\frac{N-1}{2r}\frac{x}{r}u\right|^2+\frac{(N-1)(N-3)}{4r^2}u^2\right\}dx\,d\tau\\
&\leqslant 0. \qquad (4.11)
\end{aligned}
$$

By means of the boundary condition $u|_{\partial\Omega} = 0$ we see

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} \boldsymbol{Y} \cdot \boldsymbol{\nu}\, dS\, d\tau \\
&\quad = \frac{1}{2} \int_0^t \int_{\partial\Omega} \psi \left\{ \left(\boldsymbol{\nu} \cdot \frac{x}{r} \right) |\nabla u|^2 - 2(\boldsymbol{\nu} \cdot \nabla u) \left(\frac{x}{r} \cdot \nabla u \right) \right\} dS\, d\tau \\
&\quad = -\frac{1}{2} \int_0^t \int_{\partial\Omega} \psi \left(\boldsymbol{\nu} \cdot \frac{x}{r} \right) \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 dS\, d\tau.
\end{aligned}$$

Let Γ be a part of the boundary $\partial\Omega$ such that

$$\Gamma = \left\{ x \in \partial\Omega;\ x \cdot \boldsymbol{\nu}(x) > 0 \right\}.$$

Of course, if $\mathcal{O}$ is star-shaped with respect the origin, then $\Gamma = \emptyset$. In any way we have

$$\int_0^t \int_{\partial\Omega} \boldsymbol{Y} \cdot \boldsymbol{\nu}\, dS\, d\tau \geqslant -\frac{1}{2} \sup_{s \geqslant 0} \psi(s) \int_0^t \int_{\Gamma} \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 dS\, d\tau. \tag{4.12}$$

On the other hand, there exists a constant $C_\psi > 0$ such that

$$\int_\Omega |X(t,x)|\, dx \leqslant C_\psi \|u(0)\|_E^2 \quad \text{for all } t \geqslant 0. \tag{4.13}$$

In fact, by the Schwarz inequality we have

$$\int_\Omega |X(t,x)|\, dx \leqslant \sup_{s \geqslant 0} \psi(s) \int_\Omega \left(u_t^2 + \left| u_r + \frac{N-1}{2r} u \right|^2 \right) dx.$$

Thus (4.13) follows if we use the following Hardy inequality

$$\int_\Omega \frac{u^2}{r^2}\, dx \leqslant C \int_\Omega u_r^2\, dx \tag{4.14}$$

and the energy identity (1.1). Hence it follows from (4.11), (4.12) and (4.13) that

$$\begin{aligned}
&\frac{1}{2} \int_0^t \int_\Omega \psi' \left\{ u_t^2 + \left| \nabla u + \frac{N-1}{2r} \frac{x}{r} u \right|^2 + \frac{(N-1)(N-3)}{4r^2} u^2 \right\} dx\, d\tau \\
&\qquad + \int_0^t \int_\Omega \psi a u_t \left(u_r + \frac{N-1}{2r} u \right) dx\, d\tau
\end{aligned}$$

$$\leqslant C_\psi \left\{ \|u(0)\|_E^2 + \int_0^t \int_\Gamma \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 dS\, d\tau \right\}. \tag{4.15}$$

Second step. Next, we shall estimate the boundary integral in the right-hand side of the estimate (4.15) by the L^2 estimate. For this, let us take the vector field $\boldsymbol{h}(x)$ in Lemma 3.1 such that

$$\boldsymbol{h}(x) = \begin{cases} \boldsymbol{\nu}(x) & \text{on } \Gamma, \\ 0 & \text{on } \widetilde{\omega}^c, \end{cases} \quad \text{and} \quad \boldsymbol{h}(x) \cdot \boldsymbol{\nu}(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^N,$$

where $\widetilde{\omega}$ is an open set in $\mathbb{R}^N$ such that $\overline{\Gamma} \subset \widetilde{\omega}$ and $\widetilde{\omega} \cap \Omega \subset \omega$. Then we see from the identity (3.1) in Lemma 3.1 that

$$\int_0^t \int_\Gamma \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 dS\, d\tau \leqslant C \left\{ \|u(0)\|_E^2 + \int_0^t \int_{\widetilde{\omega} \cap \Omega} \left(u_t^2 + |\nabla u|^2 \right) dx\, d\tau \right\}, \tag{4.16}$$

where the constant C depends on $\sup_{x \in \omega}\{|\boldsymbol{h}(x)| + |D\boldsymbol{h}(x)|\}$ with

$$D\boldsymbol{h}(x) = \left(\frac{\partial h_i}{\partial x_j}(x) \right)_{1 \leqslant i, j \leqslant N}.$$

Since $a(x) \geqslant a_0$ on ω, the kinetic local energy can be estimated as

$$\int_0^t \int_{\widetilde{\omega} \cap \Omega} u_t^2\, dx\, d\tau \leqslant a_0^{-1} \int_0^t \int_\omega a u_t^2\, dx\, d\tau \leqslant a_0^{-1} \|u(0)\|_E^2. \tag{4.17}$$

For the potential local energy we take $\eta \in C^1(\Omega)$ in Lemma 3.1 such that

$$\eta(x) = \begin{cases} 1 & \text{on } \widetilde{\omega} \cap \Omega, \\ 0 & \text{on } \omega^c, \end{cases}$$
$$0 \leqslant \eta(x) \leqslant 1 \quad \text{on } \Omega, \quad \text{and} \quad |\nabla \eta| \leqslant C\sqrt{\eta} \quad \text{in } \Omega \text{ for some } C > 0.$$

Thus, it follows from the identity (3.2) that

$$\int_0^t \int_{\widetilde{\omega} \cap \Omega} |\nabla u|^2\, dx\, d\tau \leqslant C \left\{ \|u(0)\|_E^2 + \int_0^t \int_\omega \left(u_t^2 + u^2 \right) dx\, d\tau \right\}. \tag{4.18}$$

Therefore, combining (4.16) and (4.17) with (4.18), we have

$$\int_0^t \int_\Gamma \left(\frac{\partial u}{\partial \boldsymbol{\nu}} \right)^2 dS\, d\tau \leqslant C \left(\|u(0)\|_E^2 + \int_0^t \int_\Omega a u^2\, dx\, d\tau \right) \tag{4.19}$$

for all $t > 0$.

Final step. It follows from the estimates (4.15) and (4.19) for an assumed smooth solution u that the estimate (4.9) is valid for the finite energy solution u. This completes the proof of Lemma 4.3. □

Lemma 4.4. *Let a be independent of t. In addition to the assumptions in Lemma* 4.3*, if we assume that the data $\{u_0, u_1\}$ further satisfies $\|au_0 + u_1\|_{L^{2,1}} < \infty$, then there exists a constant $C_5 > 0$ such that*

$$\|u(t)\|^2 + \int_0^t \int_\Omega au^2\,dx\,d\tau \leqslant C_5\bigl\{\|\{u_0, u_1\}\|^2_{H^1\times L^2} + \|au_0 + u_1\|^2_{L^{2,1}}\bigr\} \tag{4.20}$$

for all $t > 0$.

Proof. The proof is given in the same manner as in our previous paper [4]. Setting

$$w(t,x) = \int_0^t u(\tau,x)\,d\tau,$$

we see that $w(t,x)$ belongs to $C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$ and satisfies the initial–boundary value problem

$$\begin{cases} w_{tt} - \Delta w + aw_t = au_0 + u_1, & (t,x) \in (0,\infty)\times\Omega, \\ w(0,x) = 0, \quad w_t(0,x) = u_0(x), & \text{and} \quad w|_{\partial\Omega} = 0, \end{cases} \tag{4.21}$$

where we have used $a(t,x) \equiv a(x)$. Multiplying Eq. (4.21) by w_t and using the relation $w_t = u$, we have

$$\begin{aligned} &\|u(t)\|^2 + \|\nabla w(t)\|^2 + 2\int_0^t \int_\Omega au(\tau)^2\,dx\,d\tau \\ &\quad = \|u_0\|^2 + 2\int_0^t \int_\Omega (au_0 + u_1)w_t(\tau)\,dx\,d\tau. \end{aligned} \tag{4.22}$$

Here, we see

$$\begin{aligned} 2\int_0^t \int_\Omega (au_0 + u_1)w_t(\tau)\,dx\,d\tau &= 2\int_\Omega (au_0 + u_1)w(t)\,dx \\ &\leqslant 2\biggl(\int_\Omega r^2(au_0 + u_1)^2\,dx\biggr)^{1/2}\biggl(\int_\Omega \frac{w(t)^2}{r^2}\,dx\biggr)^{1/2} \\ &\leqslant C(N)\|\nabla w(t)\|\|au_0 + u_1\|_{L^{2,1}} \\ &\leqslant \varepsilon\|\nabla w(t)\|^2 + C_\varepsilon\|au_0 + u_1\|^2_{L^{2,1}} \end{aligned} \tag{4.23}$$

for $0 < \forall\varepsilon < 1$, where we have used the Hardy inequality (4.14). Therefore, it follows from (4.22) and (4.23) that Lemma 4.4 is valid for the finite energy solution $u(t,x)$. □

Proof of Proposition 4.2 (completed). Set

$$\psi(r) = 1 - \delta^{-1}\left\{\log^{[n]}(e_n + r)\right\}^{-\delta+1},$$

where $\delta > 1$. Then we have

$$\psi'(r) = \delta^{-1}(\delta - 1)\Big\{(e_n + r)\log(e_n + r)\log^{[1]}(e_n + r)\ldots \times \log^{[n-1]}(e_n + r)\big[\log^{[n]}(e_n + r)\big]^{\delta}\Big\}^{-1},$$

and it follows that

$$r^{-1}\psi(r) \geqslant (1 - \delta^{-1})(e_n + r)^{-1} \geqslant \delta^{-1}(\delta - 1)(e_n + r)^{-1} \geqslant \psi'(r).$$

Thus, (4.8) is satisfied for this $\psi(r)$.

Noting $0 < a_1 < \delta^{-1}(1 - \delta)$ and applying this ψ to Lemmas 4.3 and 4.4, we have the desired estimate (4.6).

Since $\psi'(r)$ is decreasing in $r > 0$, it follows from Lemma 3.2 that

$$\begin{aligned}
&\frac{1}{2}\int_0^t \int_{\Omega} \psi'(r)\left\{u_t^2 + \left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 + \frac{(N-1)(N-3)}{4r^2}u^2\right\} dx\, d\tau \\
&\quad \geqslant \frac{1}{2}\psi'(R)\int_0^t \int_{\Omega(R)} \left\{u_t^2 + \left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 + \frac{(N-1)(N-3)}{4r^2}u^2\right\} dx\, d\tau \\
&\quad \geqslant \psi'(R)\int_0^t E_R(\tau)\, d\tau.
\end{aligned} \tag{4.24}$$

Thus, combining the inequality (4.24) with (4.6), we have the desired estimate (4.7). The proof of Proposition 4.2 is now complete. □

5. Proof of Theorem 1

First, let us check the integrability of aw_t^2 and $(\partial w/\partial\tilde{\boldsymbol{\nu}})^2$ in the inequality (2.2).

Lemma 5.1. *Let $N \geqslant 3$ and $w(t, x)$ be the finite energy solution to the problem* (P)$_0$. *Then we have*

$$\int_0^\infty \int_{\tilde{\Omega}} aw_t^2\, dx\, dt + \int_0^\infty \int_{\tilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right)^2 dS\, dt \leqslant C\|w(0)\|_{\tilde{E}}^2. \tag{5.1}$$

Proof. Since $\mathbb{R}^N \setminus \widetilde{\Omega}$ is star-shaped with respect to the origin, the estimate

$$\int_0^\infty \int_{\widetilde{\Omega}} a w_t^2 \, dx \, dt \leqslant C \|w(0)\|_{\widetilde{E}}^2$$

has already proven in Lemma 3.3 of [11] or Proposition 4.2 for u and a replaced by w and 0, respectively.

Next, we want to prove the integrability of $(\partial w/\partial \tilde{\boldsymbol{\nu}})^2$ over $(0,t) \times \widetilde{\Gamma}$. But, the proof is given in a parallel way to the second step in the proof of Lemma 4.3. Note that Lemma 3.1 is also valid for u, a, Ω and $\boldsymbol{\nu}$ replaced by w, 0, $\widetilde{\Omega}$ and $\tilde{\boldsymbol{\nu}}$, respectively. Let us take the vector field $\boldsymbol{h}(x)$ such that

$$\boldsymbol{h}(x) = \begin{cases} \tilde{\boldsymbol{\nu}}(x) & \text{on } \widetilde{\Gamma}, \\ 0 & \text{on } \omega_2^c, \end{cases} \quad \text{and} \quad \boldsymbol{h}(x) \cdot \tilde{\boldsymbol{\nu}}(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^N,$$

where ω_2 is an open set in $\mathbb{R}^N$ such that $\overline{\widetilde{\Gamma}} \subset \omega_2$ and $\omega_2 \cap \widetilde{\Omega} \subset \omega$. Since $\mathbb{R}^N \setminus \widetilde{\Omega}$ is star-shaped with respect to the origin, it follows from Lemma 3.1 and the inequality (4.7) in Proposition 4.2 for u and a replaced by w and 0, respectively, that

$$\begin{aligned} \int_0^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right)^2 dS \, dt &\leqslant C \left\{ \|w(0)\|_{\widetilde{E}}^2 + \int_0^\infty \int_{\omega_2 \cap \widetilde{\Omega}} \left(w_t^2 + |\nabla w|^2\right) dx \, dt \right\} \\ &\leqslant C \|w(0)\|_{\widetilde{E}}^2. \end{aligned} \tag{5.2}$$

This ends the proof of Lemma 5.1. □

By virtue of Lemma 5.1 we can choose $\sigma > 0$ so that

$$\begin{aligned} &\int_\sigma^\infty \int_{\widetilde{\Omega}} a w_t^2 \, dx \, dt + \frac{C_0 \{\|u_t(0)\|_{\widetilde{E}} + \|u(0)\|_{\widetilde{E}}\}^2}{\|u(0)\|_{\widetilde{E}}^2} \int_\sigma^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right)^2 dS \, dt \\ &\quad < 2\|u(0)\|_{\widetilde{E}}^2, \end{aligned} \tag{5.3}$$

and we fix it, where C_0 is a constant depending only on ω but independent of σ, which will be precisely determined later. As is mentioned in the statement of Theorem 1, we let $\widetilde{w}(\sigma,x)$ be a zero extension of $w(\sigma,x)$ in Ω. Then $\widetilde{w}(\sigma,x)$ satisfies $\|\widetilde{w}(\sigma)\|_E < \infty$. For this $\widetilde{w}(\sigma,x)$, let $u^{(\sigma)}(t,x)$ be a finite energy solution to the problem (P) with the initial condition replaced by $\{u^{(\sigma)}(0,x), u_t^{(\sigma)}(0,x)\} = \{\widetilde{w}(\sigma,x), \widetilde{w}_t(\sigma,x)\}$.

Next, we must check the time integrability of $\|u_t^{(\sigma)}(t)\|_{L^2(\widetilde{\Gamma})}^2$ to be defined the inner-product in the energy space, which is the heart in this paper.

Proposition 5.2. *Let $u^{(\sigma)}(t,x)$ be as above. Then there exists a constant $C_0 > 0$ depending only on ω but independent of σ such that*

$$\int_0^\infty \int_{\widetilde{\Gamma}} \bigl|u_t^{(\sigma)}\bigr|^2 \, dS\, dt \leqslant C_0 \bigl\{\|u(0)\|_{\widetilde{E}} + \|u_t(0)\|_{\widetilde{E}}\bigr\}^2, \tag{5.4}$$

where we set

$$\|u_t(0)\|_{\widetilde{E}}^2 = \frac{1}{2} \int_{\widetilde{\Omega}} \bigl\{|u_{tt}(0)|^2 + |\nabla u_t(0)|^2\bigr\}\, dx.$$

Proof. Let $u(t,x)$ be the $X^2(\Omega)$-solution to the problem (P). For the proof, we first prove that there exists a constant $C_0 > 0$ depending only on ω such that

$$\int_0^t \int_{\widetilde{\Gamma}} u_t^2 \, dS\, d\tau \leqslant C_0 \bigl\{\|u(0)\|_E + \|u_t(0)\|_E\bigr\}^2 \tag{5.5}$$

for all $t > 0$.

Let ω_2 and ω_3 be open sets in $\mathbb{R}^N$ such that $\overline{\widetilde{\Gamma}} \subset \omega_2 \subset\subset \omega_3$ and $\omega_3 \cap \widetilde{\Omega} \subset \omega$, and $\psi(x)$ the C^∞ cut-off function as

$$\psi(x) = \begin{cases} 1 & \text{on } \omega_2, \\ 0 & \text{on } \omega_3^c, \end{cases} \quad \text{and} \quad 0 \leqslant \psi(x) \leqslant 1 \quad \text{for } x \in \mathbb{R}^N.$$

Then, by the trace theorem, the energy identity (1.1), $a(t,x) \geqslant a_0$ in $[0,\infty) \times \omega$ and the estimate (4.1) in Proposition 4.1, we have

$$\begin{aligned}
\int_0^t \int_{\widetilde{\Gamma}} u_t^2 \, dS\, d\tau &\leqslant \int_0^t \int_{\partial\widetilde{\Omega}} |\psi u_t|^2 \, dS\, d\tau \\
&\leqslant C \int_0^t \int_{\omega_3 \cap \widetilde{\Omega}} \bigl\{|\psi u_t|^2 + |\nabla(\psi u_t)|^2\bigr\} dx\, d\tau \\
&\leqslant C_\psi \int_0^t \int_{\omega_3 \cap \widetilde{\Omega}} \bigl(u_t^2 + |\nabla u_t|^2\bigr)\, dx\, d\tau \\
&\leqslant C_\psi a_0^{-1} \int_0^t \int_{\Omega} a\bigl(u_t^2 + |\nabla u_t|^2\bigr)\, dx\, d\tau, \\
&\leqslant C_0 \bigl\{\|u(0)\|_E + \|u_t(0)\|_E\bigr\}^2,
\end{aligned}$$

where C_0 is a constant depending on $\sup_{x\in\omega_3}\{|\psi(x)|+|\nabla\psi(x)|\}$. Therefore, the inequality (5.5) is valid.

Next, let us consider the approximate solutions $\{u^{(\sigma)}_{(j)}(t,x)\}_{j=1}^{\infty}$ of $u^{(\sigma)}(t,x)$ as follows. Let $\{w^{(j)}(\sigma,\cdot), w^{(j)}_t(\sigma,\cdot)\}_{j=1}^{\infty}\subset C_0^{\infty}(\widetilde{\Omega})\times C_0^{\infty}(\widetilde{\Omega})$ be a sequence and $\widetilde{w}^{(j)}(\sigma,x)$ a zero extension of $w^{(j)}(\sigma,x)$ to Ω such that

$$\begin{aligned}
&\bigl\{w^{(j)}(\sigma,\cdot), w^{(j)}_t(\sigma,\cdot)\bigr\}\to\bigl\{w(\sigma,\cdot), w_t(\sigma,\cdot)\bigr\}\\
&\quad\text{in } \bigl[H^2(\widetilde{\Omega})\cap H^1_0(\widetilde{\Omega})\bigr]\times H^1_0(\widetilde{\Omega}),\\
&\bigl\{\widetilde{w}^{(j)}(\sigma,\cdot), \widetilde{w}^{(j)}_t(\sigma,\cdot)\bigr\}\to\bigl\{\widetilde{w}(\sigma,\cdot), \widetilde{w}_t(\sigma,\cdot)\bigr\}\\
&\quad\text{in } H^1_0(\Omega)\times L^2(\Omega)
\end{aligned}$$

as $j\to\infty$. Here we note that since $\|w(\sigma)\|_{\widetilde{E}}$ and $\|w_t(\sigma)\|_{\widetilde{E}}$ are conservative, it follows that

$$\begin{aligned}
&\bigl\|w^{(j)}(\sigma)\bigr\|_{\widetilde{E}}\to\|w(\sigma)\|_{\widetilde{E}}=\|w(0)\|_{\widetilde{E}}=\|u(0)\|_{\widetilde{E}},\\
&\bigl\|w^{(j)}_t(\sigma)\bigr\|_{\widetilde{E}}\to\|w_t(\sigma)\|_{\widetilde{E}}=\|w_t(0)\|_{\widetilde{E}}=\|u_t(0)\|_{\widetilde{E}}
\end{aligned}$$

as $j\to\infty$, which implies that there exists a constant $C>0$, independent of j, such that

$$\bigl\|w^{(j)}(\sigma)\bigr\|_{\widetilde{E}}\leqslant C\|u(0)\|_{\widetilde{E}}\quad\text{and}\quad\bigl\|w^{(j)}_t(\sigma)\bigr\|_{\widetilde{E}}\leqslant C\|u_t(0)\|_{\widetilde{E}}\tag{5.6}$$

for sufficiently large j. Moreover, we consider the problem (P) with the initial condition $\{u_0,u_1\}$ replaced by $\{\widetilde{w}^{(j)}(\sigma),\widetilde{w}^{(j)}_t(\sigma)\}$ and let $u^{(\sigma)}_{(j)}(t,x)$ be a solution to this problem. We see from the energy identity and (5.6) that there exists a constant $C>0$, independent of j, such that

$$\bigl\|u^{(\sigma)}_{(j)}(t)\bigr\|_E\leqslant\bigl\|\widetilde{w}^{(j)}(\sigma)\bigr\|_E=\bigl\|w^{(j)}(\sigma)\bigr\|_{\widetilde{E}}\leqslant C\|u(0)\|_{\widetilde{E}},\tag{5.7}$$

$$\bigl\|\widetilde{w}^{(j)}_t(\sigma)\bigr\|_E=\bigl\|w^{(j)}_t(\sigma)\bigr\|_{\widetilde{E}}\leqslant C\|u_t(0)\|_{\widetilde{E}}\tag{5.8}$$

for sufficiently large j. Then it follows from (5.5), (5.7) and (5.8) that

$$\begin{aligned}
\int_0^T\int_{\widetilde{\Gamma}}\bigl|u^{(\sigma)}_{(j),t}(\tau)\bigr|^2\,dS\,d\tau&\leqslant C_0\bigl\{\bigl\|\widetilde{w}^{(j)}(\sigma)\bigr\|_E+\bigl\|\widetilde{w}^{(j)}_t(\sigma)\bigr\|_E\bigr\}^2\\
&\leqslant C_0\bigl\{\|u(0)\|_{\widetilde{E}}+\|u_t(0)\|_{\widetilde{E}}\bigr\}^2
\end{aligned}\tag{5.9}$$

for any fixed number $T>0$, which implies that

$$u^{(\sigma)}_{(j),t}\to u^{(\sigma)}_t\quad\text{weakly in } L^2\bigl((0,T)\times\widetilde{\Gamma}\bigr)$$

as $j\to\infty$. Thus, we can take a limiting procedure in the estimate (5.9) along a subsequence so that the limit function $u^{(\sigma)}$ is, in fact, the finite energy solution to the problem (P) with the initial condition $\{\widetilde{w}(\sigma),\widetilde{w}_t(\sigma)\}$ and satisfies

$$\int_0^T \int_{\widetilde{\Gamma}} \left|u_t^{(\sigma)}(\tau)\right|^2 dS\, d\tau \leqslant \liminf_{j\to\infty} \int_0^T \int_{\widetilde{\Gamma}} \left|u_{(j),t}^{(\sigma)}(\tau)\right|^2 dS\, d\tau$$
$$\leqslant C_0\left\{\|u(0)\|_{\widetilde{E}} + \|u_t(0)\|_{\widetilde{E}}\right\}^2.$$

The proof of Proposition 5.2 is now finished. □

Our argument is based on the following identity of the inner-product in the energy space over $\widetilde{\Omega}$.

Lemma 5.3. *Let $u^{(\sigma)}(t,x)$ and $w(t,x)$ be as above. Then we have*

$$2\left(u^{(\sigma)}(t), w(t+\sigma)\right)_{\widetilde{E}} + \int_0^t \int_{\widetilde{\Omega}} a u_t^{(\sigma)}(\tau) w_t(\tau+\sigma)\, dx\, d\tau$$
$$= 2\|u(0)\|_{\widetilde{E}}^2 + \int_0^t \int_{\Gamma_1} u_t^{(\sigma)}(\tau) \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}(\tau+\sigma)\, dS\, d\tau \tag{5.10}$$

for all $t>0$, where $(\cdot\,,\cdot)_{\widetilde{E}}$ is the inner-product in the energy space $\widetilde{E}$:

$$\left(u(t), w(t)\right)_{\widetilde{E}} = \frac{1}{2}\int_{\widetilde{\Omega}} (u_t w_t + \nabla u \cdot \nabla w)\, dx. \tag{5.11}$$

Proof. As in the proof of Proposition 5.2, we may prove the assertion for the approximated solutions $\{u_{(j)}^{(\sigma)}(t,x)\}$. We note that $u_{(j),t}^{(\sigma)} = 0$ on $\partial\widetilde{\Omega} \setminus \widetilde{\Gamma}$ and $w_t = 0$ on $\partial\widetilde{\Omega}$ for all $t>0$. Differentiating $(u_{(j)}^{(\sigma)}(t), w(t+\sigma))_{\widetilde{E}}$ and using the equations in (P) and $(\mathrm{P})_0$, respectively, we see

$$2\frac{d}{dt}\left(u_{(j)}^{(\sigma)}(t), w(t+\sigma)\right)_{\widetilde{E}}$$
$$= -\int_{\widetilde{\Omega}} a u_{(j),t}^{(\sigma)} w_t\, dx + \int_{\partial\widetilde{\Omega}} \left(\frac{\partial u_{(j)}^{(\sigma)}}{\partial \tilde{\boldsymbol{\nu}}} w_t + u_{(j),t}^{(\sigma)} \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right) dS$$
$$= -\int_{\widetilde{\Omega}} a u_{(j),t}^{(\sigma)} w_t\, dx + \int_{\widetilde{\Gamma}} u_{(j),t}^{(\sigma)} \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\, dS. \tag{5.12}$$

Integrating (5.12) over $(0,t)$, we have the identity

$$2\big(u_{(j)}^{(\sigma)}(t), w(t+\sigma)\big)_{\widetilde{E}} + \int_0^t \int_{\widetilde{\Omega}} a u_{(j),t}^{(\sigma)} w_t \, dx \, d\tau$$

$$= 2\big(w^{(j)}(\sigma), w(\sigma)\big)_{\widetilde{E}} + \int_0^t \int_{\widetilde{\Gamma}} u_{(j),t}^{(\sigma)} \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}} \, dS \, d\tau. \tag{5.13}$$

Thus, noting that

$$\big(w(\sigma), w(\sigma)\big)_{\widetilde{E}} = \|u(0)\|_{\widetilde{E}}^2,$$

we can take the limiting procedure. □

Proof of Theorem 1 (completed). First we shall prove the assertion (i). Now we suppose that $\|u^{(\sigma)}(t)\|_E \to 0$ as $t \to \infty$ and lead to a contradiction. Letting $t \to \infty$ in (5.10), we have

$$\int_0^\infty \int_{\widetilde{\Omega}} a u_t^{(\sigma)}(t) w_t(t+\sigma) \, dx \, dt$$

$$= 2\|u(0)\|_{\widetilde{E}}^2 + \int_0^\infty \int_{\widetilde{\Gamma}} u_t^{(\sigma)}(t) \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}(t+\sigma) \, dS \, dt. \tag{5.14}$$

Moreover, noting

$$\big\|u^{(\sigma)}(0)\big\|_E^2 = \|\widetilde{w}(\sigma)\|_E^2 = \|w(\sigma)\|_{\widetilde{E}}^2 = \|u(0)\|_{\widetilde{E}}^2,$$

we see from the energy identity (1.1) that

$$\int_0^\infty \int_{\widetilde{\Omega}} a \big|u_t^{(\sigma)}(t)\big|^2 \, dx \, dt \leqslant \|u(0)\|_{\widetilde{E}}^2. \tag{5.15}$$

On the other hand, it follows from the Schwarz inequality and Proposition 5.2 that

$$\left| \int_0^\infty \int_{\widetilde{\Gamma}} u_t^{(\sigma)}(t) \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}(t+\sigma) \, dS \, dt \right|$$

$$\leqslant C_0^{1/2} \big\{ \|u(0)\|_{\widetilde{E}} + \|u_t(0)\|_{\widetilde{E}} \big\} \left(\int_\sigma^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}(t) \right)^2 dS \, dt \right)^{1/2}. \tag{5.16}$$

Applying the Schwarz inequality to the left-hand side of the identity (5.14) and using (5.15) and (5.16), we conclude that

$$\int_\sigma^\infty \int_{\widetilde{\Omega}} a w_t^2 \, dx \, dt + \frac{C_0 \{\|u_t(0)\|_{\widetilde{E}} + \|u(0)\|_{\widetilde{E}}\}^2}{\|u(0)\|_{\widetilde{E}}^2} \int_\sigma^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}}\right)^2 dS \, dt$$
$$\geqslant 2\|u(0)\|_{\widetilde{E}}^2,$$

which contradicts the inequality (5.3). Thus $\|u^{(\sigma)}(t)\|_E$ never decays as $t \to \infty$. The proof of (i) is complete.

Next, we shall prove (ii). Let $U_0(t)$, $t \in \mathbb{R}$, be the unitary group in the energy space $\widetilde{E}$ which represents the solution $w(t, x)$ to the problem $(\mathrm{P})_0$ with the data $\boldsymbol{f} \equiv \{w_0, w_1\} \in \widetilde{E}$:

$$\{w(t), w_t(t)\} = U_0(t)\boldsymbol{f}.$$

Then it follows from Lemma 5.3 for $u^{(\sigma)}(t)$ and $w(t+\sigma)$ replaced by $u(t)$ and $w(t)$, respectively, that

$$\big(U_0(-t)\boldsymbol{u}(t) - U_0(-s)\boldsymbol{u}(s), \boldsymbol{f}\big)_{\widetilde{E}}$$
$$= -\int_s^t \int_{\widetilde{\Omega}} a u_t w_t \, dx \, d\tau + \int_s^t \int_{\widetilde{\Gamma}} u_t \frac{\partial w}{\partial \tilde{\boldsymbol{\nu}}} \, dS \, d\tau$$

for any $0 \leqslant s < t$, where $\boldsymbol{u}(t)$ stands for the pair $\{u(t), u_t(t)\}$. By the Schwarz inequality and Lemma 5.1 we have

$$\big|\big(U_0(-t)\boldsymbol{u}(t) - U_0(-s)\boldsymbol{u}(s), \boldsymbol{f}\big)_{\widetilde{E}}\big|$$
$$\leqslant C\|\boldsymbol{f}\|_{\widetilde{E}} \left(\int_s^t \int_{\widetilde{\Omega}} a u_t^2 \, dx \, d\tau + \int_s^t \int_{\widetilde{\Gamma}} u_t^2 \, dS \, d\tau\right)^{1/2},$$

which implies from the energy identity (1.1) and the estimate (5.5) in Proposition 5.2 that

$$\big\|U_0(-t)\boldsymbol{u}(t) - U_0(-s)\boldsymbol{u}(s)\big\|_{\widetilde{E}} \to 0 \quad \text{as } s, t \to \infty,$$

and $U_0(-t)\boldsymbol{u}(t)$ converges in $\widetilde{E}$ as $t \to \infty$. Put

$$\boldsymbol{f}^+ \equiv \{w_0^+, w_1^+\} = s - \lim_{t\to\infty} U_0(-t)\boldsymbol{u}(t).$$

Then $\boldsymbol{f}^+ \in \widetilde{E}$ and we have

$$\big\|\boldsymbol{u}(t) - U_0(t)\boldsymbol{f}^+\big\|_{\widetilde{E}} = \big\|U_0(-t)\boldsymbol{u}(t) - \boldsymbol{f}^+\big\|_{\widetilde{E}} \to 0 \quad \text{as } t \to \infty.$$

The proof of (ii) is complete. □

6. Proof of Theorem 2

In this section let us derive the local energy decay. We will need the following estimate in the course of the proof.

Lemma 6.1. *Let* $N \geqslant 3$ *and* $0 < a_1 < \delta^{-1}(\delta - 1)$. *Assume that the initial data* $\{u_0, u_1\}$ *has a compact support as in* (2.4). *If* $u(t,x)$ *is the finite energy solution to the problem* (P), *then we have*

$$\int_0^\infty \int_\Omega ar|\nabla u|^2\,dx\,dt \leqslant C(R)\|u(0)\|_E^2. \tag{6.1}$$

Proof. Since we have

$$\begin{aligned}\left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 &= |\nabla u|^2 + \frac{N-1}{r}(\tilde{x}\cdot\nabla u)u + \frac{(N-1)^2}{4r^2}u^2 \\ &\geqslant \frac{3}{4}|\nabla u|^2 - \frac{3(N-1)^2}{4r^2}u^2,\end{aligned}$$

it follows that

$$\frac{3}{4}|\nabla u|^2 \leqslant \left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 + \frac{3(N-1)^2}{4r^2}u^2,$$

which implies

$$\begin{aligned}\int_0^\infty \int_\Omega ar|\nabla u|^2\,dx\,dt &\leqslant C\int_0^\infty \int_\Omega ar\left(\left|\nabla u + \frac{N-1}{2r}\frac{x}{r}u\right|^2 + \frac{u^2}{r^2}\right)dx\,dt \\ &\leqslant C(R)\left\{\|u(0)\|_E^2 + \int_0^\infty \int_\Omega a\frac{u^2}{r}\,dx\,dt\right\},\end{aligned} \tag{6.2}$$

where we have used Proposition 4.2 and $\|au_0 + u_1\|_{L^{2,1}} \leqslant C(R)\|u(0)\|_E$. In the case when $N \geqslant 4$, it follows from the estimate (6.2) and Proposition 4.2 that the estimate (6.1) is clearly valid, but a slight modification must be needed for $N = 3$. In any way, we have the estimate (6.1). In fact, because of $\partial\Omega \subset B_{\rho_0}$, there exists a constant $C_* > 0$ such that $r \geqslant C_*\rho_0$. Hence it follows from Lemma 4.4 that

$$\int_0^\infty \int_\Omega a\frac{u^2}{r}\,dx\,dt \leqslant \frac{1}{C_*^2\rho_0}\int_0^\infty \int_\Omega au^2\,dx\,dt \leqslant C(R)\|u(0)\|_E^2. \tag{6.3}$$

Thus we conclude from the estimates (6.2) and (6.3) that the estimate (6.1) can be obtained. This ends the proof of Lemma 6.1. □

For the proof of Theorem 2, we may proceed the argument for an assumed smooth solution u. Obviously we have

$$E_R(t) \leqslant \|u(0)\|_E^2 \tag{6.4}$$

for all $t \geqslant 0$, where we set

$$E_R(t) = \frac{1}{2} \int\limits_{\Omega(R)} \left\{|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right\} dx.$$

Let $t > T_0$ for any fixed $T_0 > 0$. Multiplying the equation by $tu_t + x \cdot \nabla u + ((N-1)/2)u$, we have

$$\begin{aligned} t\|u(t)\|_E^2 &+ \big(x \cdot \nabla u(t), u_t(t)\big) + \frac{N-1}{2}\big(u(t), u_t(t)\big) + \int\limits_0^t \int\limits_\Omega \tau a u_t^2 \, dx\, d\tau \\ &+ \frac{N-1}{2} \int\limits_\Omega a u^2 \, dx + \int\limits_0^t \int\limits_\Omega a(x \cdot \nabla u) u_t \, dx\, d\tau \\ &= (x \cdot \nabla u_0, u_1) + \frac{N-1}{2}(u_0, u_1) + \frac{N-1}{2} \int\limits_\Omega a u_0^2 \, dx \\ &+ \int\limits_0^t \int\limits_{\partial\Omega} \{x \cdot \nu(x)\} \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)^2 dx\, d\tau. \end{aligned} \tag{6.5}$$

Since $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B_R$, the terms independent of t in the identity (6.5) are bounded by $C(R)\|u(0)\|_E^2$ for some $C(R) > 0$. We must estimate the other terms in the identity (6.5).

It follows from Lemma 4.4 and the energy identity (1.1) that

$$\frac{N-1}{2}\left|\big(u(t), u_t(t)\big)\right| \leqslant C\left\{\|u(0)\|_E^2 + \|u(t)\|^2\right\} \leqslant C(R)\|u(0)\|_E^2. \tag{6.6}$$

As in the technique in the proof of Lemma 4.3, we can estimate the boundary integral term as

$$\begin{aligned} \int\limits_0^t \int\limits_{\partial\Omega} \left\{x \cdot \boldsymbol{\nu}(x)\right\} \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)^2 dS\, d\tau &\leqslant \sup_{x \in \Gamma} |x| \int\limits_0^t \int\limits_\Gamma \left\{\tilde{x} \cdot \boldsymbol{\nu}(x)\right\} \left(\frac{\partial u}{\partial \boldsymbol{\nu}}\right)^2 dS\, d\tau \\ &\leqslant C(R)\|u(0)\|_E^2. \end{aligned} \tag{6.7}$$

Further, we see from Lemma 6.1, our assumption (2.6) and Proposition 4.2 that

$$\left|\int_0^t\int_\Omega a(x\cdot\nabla u)u_t\,dx\,d\tau\right| \leqslant \frac{1}{2}\int_0^t\int_\Omega ar|\nabla u|^2\,dx\,d\tau+\frac{1}{2}\int_0^t\int_\Omega aru_t^2\,dx\,d\tau \leqslant C(R)\|u(0)\|_E^2. \tag{6.8}$$

Hence, it follows from (6.5)–(6.8) that

$$t\|u(t)\|_E^2 \leqslant C(R)\|u(0)\|_E^2+\left|\left(x\cdot\nabla u(t),u_t(t)\right)\right|. \tag{6.9}$$

Finally, since $\operatorname{supp} u(t,\cdot)\subset B_{R+t}$, we have

$$\begin{aligned}
&\left|\left(x\cdot\nabla u(t),u_t(t)\right)\right| \\
&\quad\leqslant \int_{\Omega(R)}|x||\nabla u(t)||u_t(t)|\,dx+\int_{B_{R+t}\setminus B_R}|x||\nabla u(t)||u_t(t)|\,dx \\
&\quad\leqslant RE_R(t)+\frac{R+t}{2}\int_{B_{R+t}\setminus B_R}\left\{u_t(t)^2+|\nabla u(t)|^2\right\}dx \\
&\quad\leqslant C(R)\|u(0)\|_E^2+\frac{t}{2}\int_{B_{R+t}\setminus B_R}\left\{u_t(t)^2+|\nabla u(t)|^2\right\}dx
\end{aligned} \tag{6.10}$$

for all $t>T_0$. Therefore, combining the estimate (6.9) with (6.10), we arrive at the inequality

$$tE_R(t)\leqslant C(R)\|u(0)\|_E^2 \tag{6.11}$$

for all $t>T_0$. Thus, we conclude from the estimates (6.4) and (6.11) that

$$E_R(t)\leqslant C(R)\|u(0)\|_E^2(1+t)^{-1}$$

for all $t\geqslant 0$. The proof of Theorem 2 is now complete. □

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